Theorems in Linear Algebra, Lay, 3rd ed.

Solutions of Linear Systems: A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

Augmented Matrices: If the augmented matrices of two linear systems are *row equivalent*, then the two systems have the same solution set, i.e., they are *equivalent*.

Theorem 1 (Chap. 1): Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2 (Chap. 1): Existence and Uniqueness of Solutions to a SOLE:

- A SOLE is consistent if and only if the rightmost column of its augmented matrix is *not* a pivot column.
- A consistent SOLE has a *unique* solution iff there are no free variables.
- A consistent SOLE has *infinitely many* solutions iff there is at least one free variable.

Algebraic Properties of R^n: For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in R^n and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	(commutative)	(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	(distributive)
(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	(associative)	(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$	(distributive)
(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$	(identity)	(vii) $c(d\mathbf{u})=(cd)(\mathbf{u})$	(associative)
(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$	(inverse)	(viii) $1\mathbf{u}=\mathbf{u}$	(identity)

Theorem 3 (Chap. 1): If A is an $m\times n$ matrix, with columns $\mathbf{a}_1,\ldots,\mathbf{a}_n$ (in R^m), and if $\mathbf{b}$ is in R^m, the matrix equation

$$A\mathbf{x}=\mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1+x_2\mathbf{a}_2+\ldots+x_n\mathbf{a}_n=\mathbf{b}$$

which, in turn, has the same solution set as the SOLE whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}]$$

Existence of Solutions: The equation $A\mathbf{x}=\mathbf{b}$ has a solution iff $\mathbf{b}$ is a linear combination of the columns of A.

Theorem 4 (Chap. 1): Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in R^m, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in R^m is a linear combination of the columns of A.

c. The columns of A span R^m.

d. A has a pivot position in every row.

Theorem 5 (Chap. 1): Linearity of Matrix-Vector Product: If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Homogeneous Equation - Nontrivial Solution: The homogeneous equation $A\mathbf{x} = 0$ has a nontrivial solution iff the equation has at least one free variable.

Theorem 6 (Chap. 1): Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = 0$.

Linearly Independent Columns: The columns of a matrix A are linearly independent iff the equation $A\mathbf{x} = 0$ has *only* the trivial solution.

Linear Independence for One Vector: A set containing only one vector $\mathbf{v}$ is linearly independent iff $\mathbf{v} \neq 0$.

Linear Independence for Two Vectors: A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent iff at least one of the vectors is a multiple of the other. The set is linearly independent iff neither of the vectors is a multiple of the other.

Theorem 7 (Chap. 1): Characterization of Linearly Dependent Sets: An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Theorem 8 (Chap. 1): More Vectors than Entries: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in R^n is linearly dependent if $p > n$.

Theorem 9 (Chap. 1): Set Containing the Zero Vector: If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in R^n contains the zero vector, then the set is linearly dependent.

Properties of Linear Transformations: If T is a linear transformation, then

(i) $T(\mathbf{0}) = \mathbf{0}$, and

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d. More generally
$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)$.

Theorem 10 (Chap 1.): Matrix of a Linear Transformation: Let $T : R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } R^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in R^n:

$$A = [\, T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n) \,]$$

The matrix A is called the **standard matrix for T**.

Theorem 11 (Chap. 1): One-to-One: Let $T : R^n \to R^m$ be a linear transformation. Then T is one-to-one iff the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem 12 (Chap. 1): One-to-One, Onto, Standard Matrix: Let $T : R^n \to R^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps R^n onto R^m iff the columns of A span R^m;

b. T is one-to-one iff the columns of A are linearly independent.