Math 5 *Linear Algebra Theorems*

Theorems in Linear Algebra, Lay, 3rd ed.

Solutions of Linear Systems: A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

Augmented Matrices: If the augmented matrices of two linear systems are *row equivalent*, then the two systems have the same solution set, i.e., they are *equivalent*.

Theorem 1 (Chap. 1): Uniqueness of the Reduced Echelon Form: Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2 (Chap. 1): Existence and Uniqueness of Solutions to a SOLE:

- A SOLE is consistent if and only if the rightmost column of its augmented matrix is *not* a pivot column.
- A consistent SOLE has a *unique* solution iff there are no free variables.
- A consistent SOLE has *infinitely many* solutions iff there is at least one free variable.

Algebraic Properties of R^n: For all u, v, w in R^n and all scalars c and d:

(i) $u + v = v + u$	(commutative)	(v) $c(u + v) = cu + cv$	(distributive)
(ii) $(u + v) + w = u + (v + w)$	(associative)	(vi) $(c + d)u = cu + du$	(distributive)
(iii) $u + 0 = 0 + u = u$	(identity)	(vii) $c(du) = (cd)(u)$	(associative)
(iv) $u + (-u) = -u + u = 0$, where $-u$ denotes $(-1)u$	(inverse)	(viii) $1u = u$	(identity)

Theorem 3 (Chap. 1): If A is an $m \times n$ matrix, with columns $a_1, \dots, a_n$ (in R^m), and if b is in R^m, the matrix equation

$$A\,x = b$$

has the same solution set as the vector equation

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$$

which, in turn, has the same solution set as the SOLE whose augmented matrix is

$$[a_1 \quad a_2 \quad \dots \quad a_n \quad b]$$

Existence of Solutions: The equation $A\,x = b$ has a solution iff b is a linear combination of the columns of A.

Theorem 4 (Chap. 1): Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each b in R^m, the equation $A\,x = b$ has a solution.

b. Each b in R^m is a linear combination of the columns of A.

c. The columns of A span R^m.

d. A has a pivot position in every row.

Theorem 5 (Chap. 1): Linearity of Matrix-Vector Product: If A is an $m \times n$ matrix, u and v are vectors in R^n, and c is a scalar, then:

a. $A(u + v) = Au + Av$;

b. $A(cu) = c(Au)$.

Homogeneous Equation - Nontrivial Solution: The homogeneous equation $Ax = 0$ has a nontrivial solution iff the equation has at least one free variable.

Theorem 6 (Chap. 1): Suppose the equation $A\,x = b$ is consistent for some given b, and let p be a solution. Then the solution set of $A\,x = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation $Ax = 0$.

Linearly Independent Columns: The columns of a matrix A are linearly independent iff the equation $A\,x = 0$ has *only* the trivial solution.

Linear Independence for One Vector: A set containing only one vector v is linearly independent iff $v \neq 0$.

Linear Independence for Two Vectors: A set of two vectors $\{v_1, v_2\}$ is linearly dependent iff at least one of the vectors is a multiple of the other. The set is linearly independent iff neither of the vectors is a multiple of the other.

Theorem 7 (Chap. 1): Characterization of Linearly Dependent Sets: An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Theorem 8 (Chap. 1): More Vectors than Entries: If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ in R^n is linearly dependent if $p > n$.

Theorem 9 (Chap. 1): Set Containing the Zero Vector: If a set $S = \{v_1, \ldots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent.

Properties of Linear Transformations: If T is a linear transformation, then

(i) $T(0) = 0$, and

(ii) $T(cu + dv) = cT(u) + dT(v)$ for all vectors u, v in the domain of T and all scalars c, d. More generally
$T(c_1 v_1 + \ldots + c_p v_p) = c_1 T(v_1) + \ldots + c_p T(v_p)$.

Theorem 10 (Chap 1.): Matrix of a Linear Transformation: Let $T: R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \quad \text{for all } x \text{ in } R^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in R^n:

$$A = [\, T(e_1) \quad \cdots \quad T(e_n) \,]$$

The matrix A is called the **standard matrix for T**.

Theorem 11 (Chap. 1): One-to-One: Let $T: R^n \to R^m$ be a linear transformation. Then T is one-to-one iff the equation $T(x) = 0$ has only the trivial solution.

Theorem 12 (Chap. 1): One-to-One, Onto, Standard Matrix: Let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps R^n onto R^m iff the columns of A span R^m;

b. T is one-to-one iff the columns of A are linearly independent.

Theorem 1 (Chap. 2): Algebraic Properties of Matrices: Let A, B, and C be matrices of the same size, and let c and d be scalars.

(i) $A + B = B + A$ (commutative)

(ii) $(A + B) + C = A + (B + C)$ (associative)

(iii) $A + 0 = 0 + A = A$ (identity)

(v) $c(A+B) = cA + cB$ (distributive)
(vi) $(c+d)A = cA + dA$ (distributive)
(iv) $A + (-A) = -A + A = 0$, where $-A$ denotes $(-1)A$ (inverse)
(vii) $c(dA) = (cd)A$ (associative)
(viii) $1A = A$ (identity)

Row-Column (Dot Product) Rule for Computing AB: If AB is defined, then the (i, j)-entry in AB is the dot product of the ith row of A with the jth column of B.

Theorem 2 (Chap. 2): Properties of Matrix Multiplication: Let A be an $m \times n$ matrix, and let B and C have sizes for which indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B+C) = AB + AC$ (left distributive law)
c. $(B+C)A = BA + CA$ (right distributive law)
d. $c(AB) = (cA)B = A(cB)$ for any scalar c
e. $I_m A = A = AI_n$ (identity for matrix multiplication)

Theorem 3 (Chap. 2): Properties of Transpose: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar c, $(cA)^T = cA^T$

d. $(AB)^T = B^T A^T$

Uniqueness of Inverse: If A has an inverse, it is unique. We denote this inverse by A^{-1}.

Theorem 4 (Chap. 2): Inverse of a 2×2 Matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible. The quantity $ad - bc$ is called the **determinant** of A and is written det A.

Theorem 5 (Chap. 2): If A is an invertible $n \times n$ matrix, then for each b in R^n, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Theorem 6 (Chap. 2): a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

More generally, the product of any number of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Properties of Elementary Matrices:

a. If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

b. Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem 7 (Chap. 2): An $n \times n$ matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Algorithm for Finding A^{-1}: Row reduce the augmented matrix $[A\ I]$. If A is row equivalent to I, then $[A \ \ I]$ is row equivalent to $[I \ \ A^{-1}]$. Otherwise, A does not have an inverse.

Theorem 8 (Chap. 2): The Invertible Matrix Theorem: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

c^{I}. A has a pivot in every column.

c^{II}. A has a pivot in every row.

d. The equation $Ax = 0$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $x \mapsto Ax$ is one-to-one.

g. The equation $Ax = b$ has at least one solution for each b in R^n.

g^{I}. The SOLE $Ax = b$ is consistent for all b in R^n.

g^{II}. The equation $Ax = b$ has at most one solution for each b in R^n.

g^{III}. The SOLE $Ax = b$ has no free variables.

g^{IIII}. The equation $Ax = b$ has exactly one solution for each b in R^n.

h. The columns of A span R^n.

i. The linear transformation $x \mapsto Ax$ maps R^n onto R^n.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of R^n.

n. Col $A = R^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{0\}$

q^{I}. The kernel of the transformation $x \mapsto Ax$ is $\{0\}$.

r. dim Nul $A = 0$

r^{I}. The nullity of A is 0.

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

u. $(\text{Col } A)^{\perp} = \{0\}$.

v. $(\text{Nul } A)^{\perp} = R^n$.

w. Row $A = R^n$.

Invertibility: Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Theorem 9 (Chap. 2): Invertible Linear Transformation: Let $T: R^n \to R^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible iff A

is an invertible matrix. In that case, the linear transformation S given by $S(x) = A^{-1}x$ is the unique inverse function for T.

Theorem 1 (Chap. 3): Cofactor Expansion of Determinant: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Theorem 2 (Chap. 3): Triangular Matrix: If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Theorem 3 (Chap. 3): Properties of Determinants: Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then det B= det A.

b. If two rows of A are interchanged to produce B, then det $B = -$ det A.

c. If one row of A is multiplied by k to produce B, then det $B = k \cdot$ det A.

Theorem 4 (Chap. 3): Invertibility: A square matrix A is invertible iff det $A \neq 0$.

Theorem 5 (Chap. 3): Transpose: If A is an $n \times n$ matrix, then det A^T = det A.

Theorem 6 (Chap. 3): Multiplicative Property: If A and B are $n \times n$ matrices, then det AB = (det A)(det B).

Theorem 9 (Chap. 3): Area & Volume: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|$ det $A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|$ det $A|$.

Theorem 10 (Chap. 3: Linear Transformations of Area & Volume: Let $T: R^2 \rightarrow R^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in R^2, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in R^3, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

These conclusions also hold whenever S is a region in R^2 with finite area or a region in R^3 with finite volume.

Vector Space Simple Facts: For each vector u in vector space V and each scalar c,

a. $0u = 0$

b. $c0 = 0$

c. $-u = (-1)u$

Theorem 1 (Chap. 4): If $v_1, \ldots, v_p$ are in a vector space V, then Span $\{v_1, \ldots, v_p\}$ is a subspace of V.

Theorem 2 (Chap. 4): The null space of an $m \times n$ matrix A is a subspace of R^n. Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of R^n.

Theorem 3 (Chap. 4): The column space of an $m \times n$ matrix A is a subspace of R^m.

Does Col A span R^m? The column space of an $m \times n$ matrix A is all of R^m iff the equation $Ax = b$ has a solution for each b in R^m.

Theorem 4 (Chap. 4): An indexed set $\{v_1, \ldots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent iff some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Theorem 5 (Chap. 4): Spanning Set Theorem: Let $S = \{v_1, \ldots, v_p\}$ be a set in V, and let $H = \text{Span}\{v_1, \ldots, v_p\}$.

a. If one of the vectors in S— say, v_k — is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.

b. If $H \neq \{0\}$, some subset of S is a basis for H.

Elementary Row Operations: Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Theorem 6 (Chap. 4): The pivot columns of a matrix A form a basis for Col A.

Theorem 7 (Chap. 4): The Unique Representation Theorem :

Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then for each x in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \ldots + c_n b_n$$

Theorem 8 (Chap. 4): Coordinate Mapping: Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [x]_B$ is a one-to-one linear transformation from V onto R^n.

Theorem 9 (Chap. 4): If a vector space V has a basis $B = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 (Chap. 4): If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Theorem 11 (Chap. 4): Dimension of Subspace: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Theorem 12 (Chap. 4): The Basis Theorem: Let V be a p-dimensional vector space, p 1. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Number of Free & Pivot Variables: The dimension of Nul A is the number of free variables in the equation $Ax = 0$, and the dimension of Col A is the number of pivot columns in A.

Theorem 13 (Chap. 4): If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Theorem 14 (Chap. 4): The Rank Theorem: The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

Theorem 1 (Chap. 5): The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2 (Chap. 5): If $v_1, \ldots, v_r$ are eigenvectors that correspond to distinct eigenval- ues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

Characteristic Equation: A scalar λ is an eigenvalue of an n λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

Theorem 4 (Chap. 5): If n n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplici- ties).

Theorem 5 (Chap. 5): The Diagonalization Theorem: An n n matrix A is diago- nalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Theorem 6 (Chap. 5): An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem 7 (Chap. 5): Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace (geometric multiplicity) for λ_k is less than or equal to the (algebraic) multiplicity of the eigenvalue λ_k.
b. The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if and only if the dimension of the eigenspace for each λ_k equals the (algebraic) multiplicity of λ_k.

c. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for R^n.

be vectors in R^n, and let c be a scalar. Then

Theorem 1 (Chap. 6): Let u, v, and w

a. $u \cdot v = v \cdot u$ (commutative)

b. $(u + v) \cdot w = u \cdot w + v \cdot w$ (distributive)

c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$ (associative)

d. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$ (positive definite)

Properties (b) and (c) together produce:

$$(c_1 u_1 + \ldots + c_p u_p) \cdot w = c_1(u_1 \cdot w) + \ldots + c_p(u_p \cdot w)$$

Norm of Scalar Times Vector: For any scalar c and vector v

$$c\,v = |c|\,v$$

Theorem 2 (Chap. 6): The Pythagorean Theorem: Two vectors u and v are orthogonal if and only if $u + v^2 = u^2 + v^2$.

Orthogonal Complement of a Subspace: If W is a subspace of R^n, then

1. A vector x is in $W^\perp$ if and only if x is orthogonal to a set that spans W.
2. $W^\perp$ is a subspace of R^n.

Theorem 3 (Chap. 6): Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the nullspace of A, and the orthogonal complement of the column space of A is the nullspace of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

Theorem 4 (Chap. 6): If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in R^n, then S is linearly independent and hence is a basis for the subspace spanned by S.

Theorem 5 (Chap. 6: Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace W of R^n. For each y in W, the weights in the linear combination

$$y = c_1 u_1 + \ldots + c_p u_p$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \ldots, p)$$

Theorem 6 (Chap. 6): An $m \times n$ matrix U has orthonormal columns iff $U^T U = I$.

Theorem 7 (Chap. 6): Let U be an $m \times n$ matrix with orthonormal columns, and let x and y be in R^n. Then

a. $\|Ux\| = \|x\|$

b. $(Ux) \cdot (Uy) = x \cdot y$

c. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$

Theorem 8 (Chap. 6): The Orthogonal Decomposition Theorem: Let W be a subspace of R^n. Then each y in R^n can be written uniquely in the form

$$y = \hat{y} + z$$

where $\hat{y}$ is in W and z is in $W^{\perp}$. In fact, if $\{u_1, \ldots, u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \ldots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

and $z = y - \hat{y}$.

Orthogonal Projection Property: If y is in W = Span $\{u_1, \ldots, u_p\}$, then proj $_W y = y$.

Theorem 9 (Chap. 6): The Best Approximation Theorem: Let W be a subspace of R^n, y any vectore in R^n, and $\hat{y}$ the orthogonal projection of y onto W. Then $\hat{y}$ is the closest point in W to y, in the sense that

$$y - \hat{y} < y - v$$

for all v in W distinct from $\hat{y}$.

Theorem 10 (Chap. 6): If $\{u_1, \ldots, u_p\}$ is an orthonormal basis for a subspace W of R^n, then

$$\text{proj}_W\, y = (y \cdot u_1)u_1 + \ldots + (y \cdot u_p)u_p$$

If $U = [u_1 \quad u_2 \quad \cdots u_p]$, then

$$\text{proj}_w\, y = UU^T y \quad \text{for all } y \text{ in } R^n$$

Theorem 13 (Chap. 6): The set of least-squares solutions of $A\,x = b$ coincides with the nonempty set of solutions of the normal equations $A^T A\,x = A^T b$.

Theorem 14 (Chap. 6): The matrix $A^T A$ is invertibile if and only if the columns of A are linearly independent. In this case, the equation $A\,x = b$ has only one least-squares solution $\hat{x}$, and it is given by

$$\hat{x} = (A^T A)^{-1} A^T b$$

Theorem 16 (Chap. 6): The Cauchy-Schwarz Inequality: For all u, v in V,

$$|\ u, v\ | \leq u \quad v$$

Theorem 17 (Chap. 6): The Triangle Inequality: For all u, v in V,

$$u + v \quad \leq \quad u \quad + v$$

Theorem 1 (Chap. 7): If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Theorem 2 (Chap. 7): An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

Theorem 4 (Chap. 7): The Principal Axes Theorem: Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $x = P\,y$, that transforms the quadratic form $x^T A\,x$ into a quadratic form $y^T D\,y$ with no cross-product term, i.e., with D a diagonal matrix. The columns of P are called the **principal axes** of the quadratic form $x^T Ax$.

Theorem 5 (Chap. 7): Quadratic Forms and Eigenvalues: Let A be an $n \times n$ symmetric matrix. Then a quadratic form $x^T A x$ is:

a. positive definite iff the eigenvalues of A are all positive,

b. negative definite iff the eigenvalues of A are all negative, or

c. indefinite iff A has both positive and negative eigenvalues.